Basics of Probability

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Probability

We call a phenomenon **random** if we are **uncertain** about its outcome.

Probability allows us to deal with randomness, by **quantifying uncertainty** and measuring the chances of the possible outcomes.

Typically the randomness *we* have to deal with comes from the **sampling procedure**: when we observe data, their values depends on the units we randomly select.

Examples of random phenomena

- the moment it will first start raining
- the result of a football match
- tomorrow's price of a stock
- the number of tweets Trump is going to write today
- ...

The basic ingredients

We call a phenomenon “random” if we are uncertain about its outcome. It is characterize by:

- **Sample Space**: the set of all possible outcomes. Its elements are exhaustive (no possible is left out) and mutually exclusive (only one outcome can occur).
- **Event**: a subset of the sample space corresponding to one or more possible outcomes
- **Probability**: measure of how likely each element of the sample space is.

The basic ingredients - an evergreen Example

Random phenomenon: throw of a die

- **Sample Space**: all the possible outcomes

  $$\Omega = \{1, 2, 3, 4, 5, 6\}$$

- **Event**: "the die returns an even number"

  $$E = \{2, 4, 6\}$$

- **Probability**:

  $$P(E) = 1/2$$

Exercises

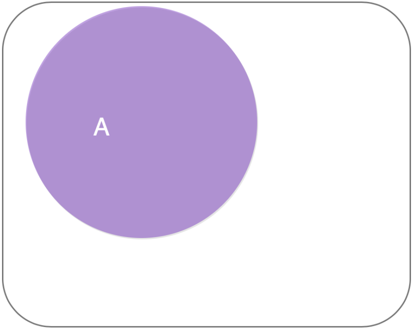
- Two coins are tossed. Note: Each coin has two possible outcomes H (Heads) and T (Tails).

1. Get the sample space.
2. Find the probability that two heads are obtained.

- A card is drawn at random from a deck of cards. Find the probability of getting a diamond.

Recap of Set theory

- **Complement** (A^c or $\bar{A}$) everything that is not in A.



Example $A =$ "the die returns an even number"

$\Rightarrow A^c =$ "the die returns an odd number"

Recap of Set theory

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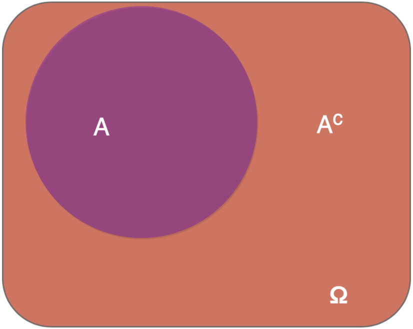
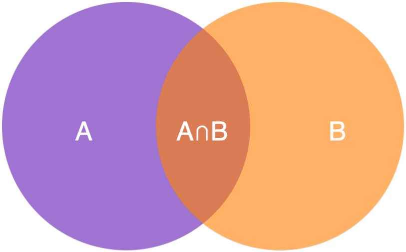


Example $A =$ "the die returns an even number", $A^c =$ "the die returns an odd number"

Recap of Set theory

- **Intersection** ($A \cap B$) given two events A, B, everything that is in *both* A and B.

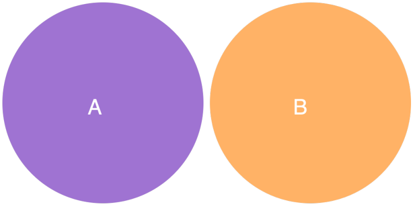


Example $A =$ “the die returns an even number”, $B =$ “the die returns a number smaller than 5”

$\Rightarrow A \cap B = \{2, 4\}$

Recap of Set theory

- **Intersection** ($A \cap B$) given two events A, B, everything that is in *both* A and B.

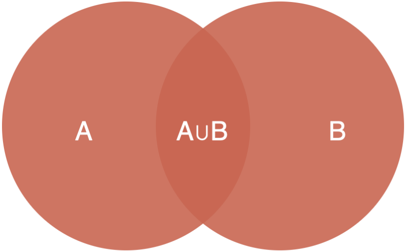


Example $A =$ "the die returns an even number", $B =$ "the die returns a 5"

$\Rightarrow A \cap B = \emptyset$ A and B are **disjoint**

Recap of Set theory

- **Union** $(A \cup B)$ given two events A, B, everything that is in *either* A, B *or in both*.



Example $A =$ "the die returns an even number", $B =$ "the die returns a 5"

$\Rightarrow A \cup B = \{2, 4, 5, 6\}$

Exercise :: challenging

Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations.

- Which is more probable?
  Linda is a bank teller.
  Linda is a bank teller and is active in the feminist movement.

Some useful relationships

Probability Axioms...

- $0 \leq P(A) \leq 1$
- $P(\Omega) = 1$
- $P(\emptyset) = 0$

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... and some trivial consequences

- $P(A^c) = 1 - P(A)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Some useful relationships

Probability Axioms...

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- $P(A^c) = 1 - P(A)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

if A, B are disjoint then $P(A \cup B) = P(A) + P(B)$

Exercises

- Which of the following is an impossible event?

1. Choosing an odd number from 1 to 10.
2. Getting an even number after rolling a single 6-sided die.
3. Choosing a white marble from a jar of 25 green marbles.
4. None of the above.

- There are 4 parents, 3 students and 6 teachers in a room. If a person is selected at random, what is the probability that it is a teacher or a student?

1. 4/13
2. 7/13
3. 9/13
4. None of the above.

How do we define probability?

something we don't really focus about

- Classical approach: Assigning probabilities based on the assumption of equally likely outcomes.

- Empirical approach: Assigning probabilities based on experimentation or historical data.

- Subjective approach: Assigning probabilities based on the assignor's judgment.

Regardless of the approach we follow, **probability is a measure of uncertainty**, i.e. it quantifies how much we do not know, hence it strongly **depends on the information available** to us.

Exercise

- Get the probability of a Italian newborn is a girl.

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1/2... isn't it?

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But, if we know that the 51.3% of Italian people are women? What does it happen to that probability?

Conditional Probability

Probability is a measure of uncertainty on the result of a random experiment, so **any additional information** on the outcome **affects it**.

Let A and B be two events, if we know that B happened, we can update the probability of A as follows:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Example: if we know that a die returned an *even* number, then the probability of observing a 3 is 0.

Independence

absence of relation between events

If knowing an event B does not affect our probability evaluation of A, then we say that A and B are **independent**:

$$P(A|B) = P(A)$$

Combining this to the definition of conditional probability we can derive the **factorization criterion**, to assess if two events are independent:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = P(A) \iff P(A \cap B) = P(A)P(B)$$

CAVEAT: The fact that two events are independent **does not mean they are disjoint**, and actually this is almost never the case. In fact for $P(A \cap B) = P(\emptyset) = 0 = P(A)P(B)$, either A or B must have probability 0.

Exercises

- Three cards are chosen at random from a deck without replacement. What is the probability of choosing an eight, a seven and a six, in order?
  - 6/35152
  - 1/2197
  - 8/16575
  - None of the above.

- A jar contains 5 red, 3 green, 2 purple and 4 yellow marbles. A marble is chosen at random from the jar. After replacing it, a second marble is chosen. What is the probability of choosing a purple and then a red marble?
  - 5/98
  - 1/2
  - 3/98
  - 2/49

Random variable

how to define it

Typically we are not interested in the single outcome itself or in the events but in a *function* of them.

A **random variable** is any function from the sample space to the real numbers.

Examples:

- toss a coin three times and **count** the number of tails
- roll two dice and **sum** the values of the faces

NB A random variable is a *number*: we can do all sorts of operations with it!

Random variables

how to characterize it

- X *random variable*: the random function (before it is observed!)
- x *realization of the random variable*: the number we get after we observe the result of the random experiment

- $\mathcal{X}$ *support of the random variable*: all the possible values assumed by X

Example:

- toss a three coins. X is the number of tails

$$\mathcal{X} = \{0, 1, 2, 3\}$$

Probability statement on a random variable can be derived from the probability on the basic events!

Distribution of a random variable

an example of how to derive it

- Toss a coin three times. X is the random variable representing the *number of Tails*

ω	$P(\omega)$	x
HHH	1/8	0
THH	1/8	1
HTH	1/8	1
HHT	1/8	1
TTH	1/8	2
THT	1/8	2
HTT	1/8	2
TTT	1/8	3

x	$p_x = P(X = x)$
0	1/8×1 = 1/8
1	1/8×3 = 3/8
2	1/8×3 = 3/8
3	1/8×1 = 1/8

The distribution of a random variable p_x is just a convenient way of summarizing single outcomes probabilities.

Exercise

- Two dice are rolled:

1. Construct the sample space. How many outcomes are there?
2. Find the probability of rolling a sum of 7.
3. Find the probability of getting a total of at least 10.
4. Find the probability of getting a odd number as the sum.

Distribution of a Discrete Random Variable:

When $\mathcal{X}$ is countable, the random variable X is said to be **discrete**, and it is characterized by:

- **Probability mass distribution**

$$p_x = P(X = x) \qquad\qquad \forall x \in \mathcal{X}$$

- **Cumulative distribution function**

$$F_X(x) = P(X \leq x) = \sum_{y \leq x} P(X = y) = \sum_{y \leq x} p_y$$

Examples:

- What is the probability of **exactly** 1 head? $P(X = 1) = p_1 = 3/8$
- What is the probability of **at most** two heads? $P(X \leq 2) = F_X(2) = p_0 + p_1 + p_2 = 7/8$

Remark

an ode to recycling

Remember: statements such as $X = 1$ or $X \leq 2$ are **events**, we can use *intersection*, *union*, *complement* and all the operations we have seen before!

Examples:

- What is the probability of **note getting** 1 head?

$$P(X \neq 1) = P((X = 1)^c) = 1 - P(X = 1) = 5/8$$

- What is the probability of **at least** 2 heads?

$$P(X \geq 2) = 1 - P(X \leq 1) = 1 - F_X(1) = 1 - (p_0 + p_1) = 4/8$$

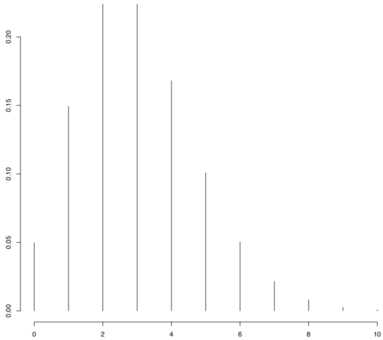
- What is the probability of 0 **or** 2 heads? (*disjoint events!*)

$$P(X = 2 \cup X = 0) = P(X = 2) + P(X = 0) = p_0 + p_2 = 4/8$$

Properties

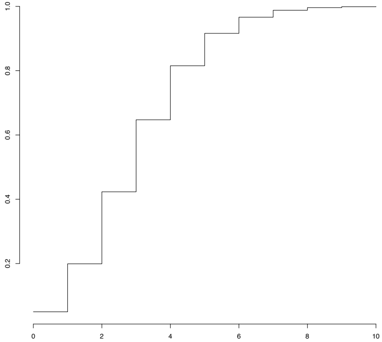
Probability mass distribution

- $p_x \geq 0$
- $p_x \leq 1$
- $\sum_x p_x = 1$



Cumulative distribution function

- $0 \leq F(x) \leq 1$
- F is *non-decreasing*
- F is *right continuous*



Exercise

- Let X be a discrete random variable with the following probability distribution

$$F(x) = \begin{cases} 0 & \text{if} \quad x < 1 \\ 1/5 & \text{if} \quad 1 \leq x < 4 \\ 3/4 & \text{if} \quad 4 \leq x < 6 \\ 1 & \text{if} \quad x \geq 6 \end{cases}$$

- Calculate the probability function.
- Calculate the following probabilities

a. $P(X=6)$
b. $P(X=5)$
c. $P(2<X<5.5)$
d. $P(0 \leq X<4)$

Distribution of a Continuous Random Variable:

When $\mathcal{X}$ is **not** countable, the random variable X is said to be **continuous**.

If $\mathcal{X}$ is not countable, it is not possible to put mass on any value $x \in \mathcal{X}$, meaning that

$$P(X = x) = 0 \qquad \forall x \in \mathcal{X}$$

- **Cumulative distribution function**

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(t) \mathrm{d}t \qquad \forall x \in \mathcal{X}$$

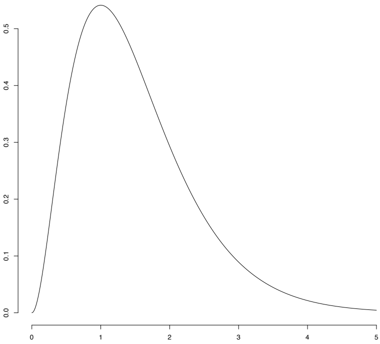
- **Probability density distribution**

$$f_X(x) = \frac{dF_X(x)}{dx} \qquad \forall x \in \mathcal{X}$$

Properties

Probability density function

- $f_X(x) \geq 0$
- $f_X(x)$ **needs not** be ≤ 1
- $\int_{-\infty}^{\infty} f_X(x)\mathrm{d}x = 1$



Cumulative distribution function

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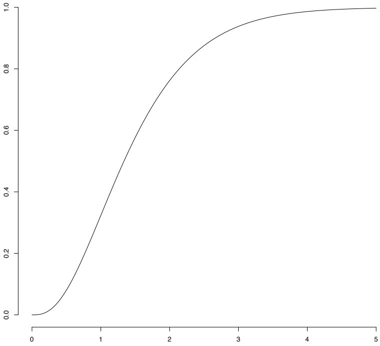


Exercise

- Let X be a continuous random variable with the following probability distribution

$$f(x) = \begin{cases} cx^2(1-x) & \text{if} \quad 0 < x \leq 1 \\ 0 & \text{if} \quad \text{otherwise} \end{cases}$$

- Determine c so that $f(x)$ is a valid pdf.

Comparison

- X discrete rv with pmf p_x
- $P(X \in A) = \sum_{x \in A} p_x$

if $A = \{x_1, \dots, x_k\}$ then

$$P(X \in A) = \sum_{i=1}^{k} p_{x_i}$$

- X continuous rv with pdf $f_X(x)$
- $P(X \in A) = \int_A f_X(x) \mathsf{d}x$

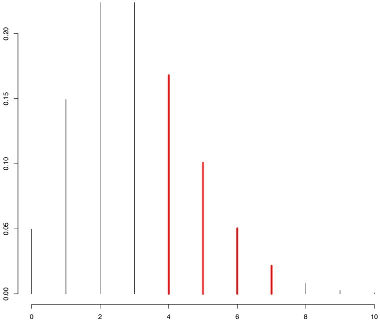
if $A = [a, b]$ then

$$P(X \in A) = \int_a^b f_X(x) \mathsf{d}x$$
$$= F_X(b) - F_X(a)$$

Comparison

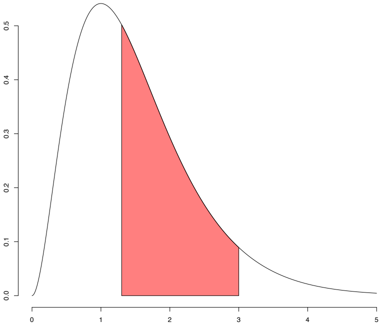
$A = \{x_1, \ldots, x_k\}$

$P(X \in A) = \sum_{i=1}^{k} p_{x_i}$



$A = [a, b]$

$P(X \in A) = F_X(b) - F_X(a)$



Summaries

The distribution of a random variables provides fully characterize it, but it may not be “immediate” to gain insights from it.

Once more we need to summarize the information contained in the distribution.

Candidates:

- *Mode*: the value that is “more likely”, i.e. the value that maximizes the density
- *Median*: the value that “splits in half” the distribution, i.e. m s.t.

$$P(X \leq m) = P(X > m) = 0.5$$

Expected Value

king of all summaries

The **Mean** or **Expected Value** is the "average" of the elements in the support of X, weighted by the probability of each outcome.

The expected value gives a rough idea of what to expect for the **average** of the observed values in a **large repetition** of the random experiment *(not what we'll observe in a single observation!)*

X discrete r.v. with p.m.f. p_x

$$\mathbb{E}[X] = \sum_{x \in \mathcal{X}} x p_x$$

X continuous r.v. with p.d.f. $f_X(x)$

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) \mathsf{d}x$$

WATCH OUT The expected value *may not exist!*

Properties

- $\mathbb{E}[c] = c$ for any constant c

  $$\mathbb{E}[\mathbb{E}[X]] = \mathbb{E}[X]$$

- $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$
- $\mathbb{E}[X - \mathbb{E}[X]] = 0$
- $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y] = 0$

The Law of the Lazy Statistician Given a continuous (respectively discrete) random variable X whose expectation exists, and a function g, then

$$\mathbb{E}[g(X)] = \int g(x) f_X(x) dx \qquad \left(\mathbb{E}[g(X)] = \sum_x g(x) p_x \right)$$

Measuring Variability

The expected value gives an idea about the **center** of the distribution, but does not account for the dispersion of the values

Example:

- Given two investment strategies with the same expected payout, we would like to choose the one with less variability

(Bad) Candidates:

- average deviation from the mean $\mathbb{E}[X - \mathbb{E}[X]]$ (**not informative**)
- average absolute deviation from the mean $\mathbb{E}|X - \mathbb{E}[X]|$ (**computationally challenging**)

Variance

queen of all summaries

The **variance** of a random variable X

$$\mathbb{V}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2]$$

tells us of **how much** the variable oscillates around the mean.

X discrete r.v. with p.m.f. p_x

$$\mathbb{V}[X] = \sum_{x \in \mathcal{X}} (x - \mathbb{E}[X])^2 p_x$$

X continuous r.v. with p.d.f. $f_X(x)$

$$\mathbb{V}[X] = \int_{-\infty}^{\infty} (x - \mathbb{E}[X])^2 f_X(x) \mathrm{d}x$$

Properties

- the variance is always **non-negative**, $\mathbb{V}[X] \geq 0$ and is 0 only when X is constant

- the square root of the variance sd$(X) = \sqrt{\mathbb{V}[X]}$ is called **standard deviation**. Roughly, ? describes how far values of the random variable fall, on the average, from the expected value of the distribution

- the variance is *insensitive to the location* of the distribution but **depends only on its scale**

$$\mathbb{V}[aX + b] = a^2 \mathbb{V}[X]$$

- a **computation-friendlier** alternative definition of the variance is:

$$\mathbb{V}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

Exercise

- Let X be a discrete random variable with the following probability distribution

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- Calculate mean and variance

Exercise

- Let X be a continuous random variable with the following probability distribution

$$f(x) = \begin{cases} cx^2(1-x) & \text{if} \quad 0 < x \leq 1 \\ 0 & \text{if} \quad \text{otherwise} \end{cases}$$

- Calculate mean and variance

Covariance

If we have 2 random variables, the **covariance** gives us a measure of association between them.

$$\begin{aligned}\mathbb{C}ov(X,Y) &= \mathbb{E}\left[(X-\mathbb{E}X)(Y-\mathbb{E}Y)\right] \\ &= \mathbb{E}[XY]-\mathbb{E}X\mathbb{E}Y\end{aligned}$$

- The sign of $\mathbb{C}ov(X,Y)$ informs on the nature of the association
- The higher $|\mathbb{C}ov(X,Y)|$, the stronger the association

Remark $\mathbb{V}[X+Y]=\mathbb{V}[X]+\mathbb{V}[Y]+2\mathbb{C}ov(X,Y)$

Independence of Random Variables

Two random variables X, Y are independent if

$$\begin{aligned} F_{X,Y}(x,y) &= P(X \le x \cap Y \le y) \\ &= P(X \le x)P(Y \le y) \\ &= F_X(x)F_Y(y) \qquad\qquad \forall x, y \in \mathbb{R} \end{aligned}$$

Intuitively if X and Y are independent, the value of one does not affect the value of the other.

Remark: if $X_1, \dots, X_n$ are independent then

- $p_{x_1,\dots,x_n} = p_{x_1} \cdot \dots \cdot p_{x_n}$
- $f_{X_1,\dots,X_n}(x_1, \dots, x_n) = f_{X_1}(x_1) \cdot \dots \cdot f_{X_n}(x_n)$

Independence of Random Variables

Factorization Criterion

$$F_{X,Y}(x,y) = F_X(x)F_Y(y) \qquad \forall x, y \in \mathbb{R}$$

If X and Y are independent then $\mathbb{E}[XY] = \mathbb{E}X\mathbb{E}Y$

As a consequence

$$\mathbb{C}ov(X,Y) = \mathbb{E}[XY] - \mathbb{E}X\mathbb{E}Y = 0$$

WATCH OUT: the converse is not true! If $\mathbb{C}ov(X,Y) = 0$, the two random variables may still be associated.

Exercise [1/2]

- Let X and Y be two random variables with marginal distribution functions

$$F_X(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 1 - exp(-x) & \text{if} \quad x \geq 0 \end{cases}$$

$$F_Y(y) = \begin{cases} 0 & \text{if} \quad y < 0 \\ 1 - exp(-y) & \text{if} \quad y \geq 0 \end{cases}$$

Exercise [2/2]

- Determine if X and Y are independent given the joint distribution function:

$$F_{X,Y}(x,y) = \begin{cases} 0 & \text{if} \quad x < 0 \quad \text{or} \quad y < 0 \\ 1 - exp(-x) - exp(-y) + exp(-x-y) & \text{if} \quad x \geq 0 \quad \text{and } y \geq 0 \end{cases}$$